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Seat No. ________________

HN-003-1162001

M. Sc. (Sem. II) Examination

April - 2023

Mathematics : CMT-2001

(Algebra-II)

Faculty Code : 003
Subject Code : 1162001

Time : $2\frac{1}{2}$ / Total Marks : **70**

Instructions : (1) All questions are compulsory.
(2) Each question carries equal marks.
(3) Figure on the right indicate allotted marks.

1 Answer any seven short questions. **7×2=14**

(i) Prove or disprove that, if $f(x)$ be a polynomial of degree 2 or 3 in a polynomial ring $F[x]$ over a field *F*, then $f(x)$ is reducible over $F \Leftrightarrow f(x)$ has a root in *F*.

(ii) Prove or disprove that, $x^3 + 3x + 2 \in \mathbb{Z}_7[x]$ is irreducible over $\mathbb{Z}_7$.

(iii) Define term : Monic polynomial and justify that every monic polynomial is a primitive polynomial.

(iv) Define terms : Extension of fields and finite field extension (finite extension).

(v) Write down the minimal polynomial of the complex number $\sqrt{2} + \sqrt{3i}$ over Q.

(vi) Write down two definitions : Perfect field and Prime field.

(vii) Write down examples of two algebraic field extenstions, which are separable extensions.

(viii) Prove or disprove that, the set of generators of an R-module need not be unique.

(ix) Let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a family of R-sub modules of an R-module M. Prove that $\bigcap_{\lambda\in\Lambda} N_\lambda$ is also an R-sub module of M.

(x) For a ring R, define R-sub module of an R-module M.

2 Attempt any two. **2×7=14**

(a) Let $E|_F$ and $K|_F$ both are finite extensions. Prove that, $K|_F$ is also finite extension.

(b) Let $p(x) \in F[x]$ be an irreducible polynomial and degree of $p(x) = n$. Let $E|_F$ be an extension such that $\alpha \in E$ and α is a root of $p(x)$. Prove that, $F[\alpha] = F(\alpha)$, $[F(\alpha):F] = n$ and $\{1, \alpha, \alpha^2, \ldots\ldots, \alpha^{n-1}\}$ is a basis of $F(\alpha)$ over F.

(c) Prove that, every finite extension is an algebraic extension.

3 Attempt followings. **2×7=14**

(1) Let $E|_F$ be a finite extension and $E = F(\alpha)$, for some $\alpha \in E$. Prove that, There are only a finite number of sub fields of E containing F, as a sub field.

(2) Let $E|_F$ be a finite extension and there are only a finite number of sub fields of E containing F, as a sub field. Prove that, $E = F(\alpha)$, for some $\alpha \in E$.

OR

3 Attempt followings. **2×7=14**

(a) Let F be a finite field. Prove that, $F^* = F - \{0\}$ is a cyclic group under multiplication.

(b) Let F be a field and $F - \{0\}$ is a cyclic group under multiplication. Prove that, F is a finite field.

4 Attempt any two. **2×7=14**

(1) Suppose following diagram of R-modules and R-homomorphisms.
$f: K \to M$, $g: M \to L$, $f': K' \to M'$, $g': M' \to L'$, $\alpha: K \to K'$, $\beta: M \to M'$ and $\gamma: L \to L'$ with $f(K) = \text{Ker } \beta = \ker g$ and $f'(K') = \text{Ker } g' = \beta(M)$. Prove that, if α, γ & f' are 1-1 then so is β.

(2) Let K be algebraically closed field and $p(x) \in K[x]$ be an irreducible polynomial over $K[x]$. Prove that degree of $p(x)$ is 1. Also deduce that, for any $f(x) \in K[x]$, with degree $f(x) \geq 1$, $f(x)$ can be split into linear factors in $K[x]$.

(3) Let F be a field and Char $F = 0$. Let n be a natural number and F contains n^{th} root of unity. Let $f(x) = x^n - a \in f[x]$ and E is the splitting field of $f(x)$ over F. Prove that, ${}^{E}|_{F}$ is a cuclic extension.

5 Attempt any two. **7×2=14**

(1) Let A, B be R-sub modules of two R-modules M and N respectively. In standard notation, prove that, $\frac{M \times N}{A \times B} \cong \frac{M}{A} \times \frac{N}{B}$.

(2) Let R be a ring with unity and M is an R-module. Prove that, M is cyclic if and only if $M \cong \frac{R}{I}$, for some left ideal I or R.

(3) Let ${}^{E}|_{F}$ be an extension and $[E:F] = 2$. Prove that, ${}^{E}|_{F}$ is a normal extension.

(4) Let ${}^{K}|_{k}$ be a cyclic extension, $[K:k] = n$ and $G({}^{K}|_{k}) = <\sigma>$. Prove that, $\beta\, \sigma(\beta)\, \sigma^2(\beta) .... \sigma^{n-1}(\beta) = 1$, for some $\beta \in K$ if and only if $\beta = \alpha . (\sigma(\alpha)^{-1})$, where $\alpha \in K^*$.